A NOTE ON THE ENERGY AND WAVELENGTH MAXIMA IN FERMI-DIRAC AND BOSE-EINSTEIN DISTRIBUTIONS*

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During recent years there have been a number of physical and astrophysical applications of quantum statistics. In the present note we shall consider the properties of the Fermi-Dirac and Bose-Einstein distribution laws which correspond to the Wien's displacement law in the case of black-body radiation. It is useful sometimes to know the value of the energy corresponding to the maximum in the energy-distribution curve of the particles in the assembly or the value of the de Broglie wavelength corresponding to the maximum in the distribution with respect to the wavelength associated with the particles. These are given in the present paper for the various cases of degeneracy and non-degeneracy.

The number of photons of black-body radiation lying within the energy range ϵ to $\epsilon + d\epsilon$ at temperature T is given by Planck's law

$$n(\epsilon)d\epsilon = \frac{8\pi\nu^2}{c^3} \cdot \frac{1}{e^{h\nu/kT} - 1} \, d\nu$$

$$= \frac{8\pi}{c^3 h^3} \cdot \frac{\epsilon^2}{e^{\epsilon/kT} - 1} \, d\epsilon,$$

so that

$$E(\epsilon)d\epsilon = \frac{8\pi\epsilon^3}{c^3 h^3 (e^{\epsilon/kT} - 1)} \, d\epsilon,$$

where $E(\epsilon)d\epsilon$ is the energy per unit volume of the photons lying in the energy range ϵ to $\epsilon + d\epsilon$.

$E(\epsilon)$ is maximum for $\epsilon = \epsilon_m$, where ϵ_m is given by

$$\frac{dE(\epsilon)}{d\epsilon} = 0,$$

which gives

$$1 = e^{\epsilon_m k/T} \left(1 - \frac{\epsilon_m}{3kT} \right)$$

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or $$\frac{\epsilon_m}{kT}=2\cdot 822.$$

If we consider the photon distribution from the standpoint of wavelength instead of energy, we have for the energy of photons in the wavelength interval λ to $\lambda+d\lambda$

$$E(\lambda)d\lambda=\frac{8\pi ch}{\lambda^5(e^{ch/\lambda kT}-1)}d\lambda$$

and $E(\lambda)$ is maximum for $\lambda=\lambda_m$, where λ_m is given by

$$\frac{dE(\lambda)}{d\lambda}=0,$$

which gives $$e^{-ch/\lambda_m kT}=1-\frac{1}{5}\frac{ch}{\lambda_m kT}$$

or $$\frac{\lambda_m kT}{ch}=\frac{1}{4\cdot 965}.$$

We shall now discuss the case of Fermi-Dirac statistics. The energy-distribution law in the completely non-relativistic case $\left(\frac{mc^2}{kT}>>1\right)$ is

$$E(\epsilon)d\epsilon=\frac{2\pi g(2m)^{\frac{3}{2}}}{h^3}\cdot\frac{\epsilon^{\frac{3}{2}}}{\frac{1}{A}e^{\epsilon/kT}+1}d\epsilon,$$

where $E(\epsilon)d\epsilon$ is the energy per unit volume of the particles lying in the energy range ϵ to $\epsilon+d\epsilon$.

Differentiating the above with respect to ϵ and equating to zero we get for ϵ_m, the value of ϵ corresponding to the maximum of the distribution curve,

$$e^{\epsilon_m/kT}\left(\frac{2}{3}\frac{\epsilon_m}{kT}-1\right)=A \qquad \ldots \quad (1)$$

or $$e^x\left(\frac{2x}{3}-1\right)=A \qquad \text{where } x=\frac{\epsilon_m}{kT}.$$

However, if we consider the energy distribution in terms of wavelength, we have

$$E(\lambda)d\lambda=\frac{2\pi gh^2}{m\lambda^6}\frac{1}{\frac{1}{A}e^{h^2/2mkT\lambda^2}+1}d\lambda.$$

Putting $$\frac{dE(\lambda)}{d\lambda}=0$$

we obtain $$e^{\frac{1}{y^2}}\left(\frac{1}{3y^2}-1\right)=A \qquad \ldots \quad (2)$$

where $$y^2 = \frac{2\lambda_m^2 mkT}{h^2}.$$

The Fermi-Dirac distribution law for the relativistic case $\left(\frac{mc^2}{kT} << 1\right)$ is

$$E(\epsilon)d\epsilon = \frac{4\pi g}{c^3 h^3} \frac{\epsilon^3}{\frac{1}{A} e^{\epsilon/kT} + 1} d\epsilon$$

and this will have a maximum at $\epsilon = \epsilon_m$, where ϵ_m is given by

$$e^x \left(\frac{x}{3} - 1\right) = A, \qquad \dots \quad (3)$$

x being equal to ϵ_m / kT.

As before, if we consider it from the wavelength standpoint, we get

$$E(\lambda)d\lambda = \frac{4\pi ghc}{\lambda^5} \frac{1}{\frac{1}{A} e^{hc/\lambda kT} + 1} d\lambda$$

which will have a maximum at $\lambda = \lambda_m$ where λ_m is given by

$$\frac{dE(\lambda)}{d\lambda} = 0$$

i.e., $$e^{\frac{1}{z}} \left(\frac{1}{5z} - 1\right) = A \qquad \dots \quad (4)$$

where $$z = \frac{\lambda_m kT}{hc}.$$

The equations (1), (2), (3) and (4) have been solved graphically and positions of the maxima due to the variation in A are shown in figures 1 and 2.

In figure (1) are shown curves for the non-degenerate case ($0 < A < 1$) and the details are as follows—

curve (*a*) shows $y = \frac{\lambda_m \sqrt{2mkT}}{h}$ against A for non-relativistic case,

curve (*b*) shows $z = \frac{\lambda_m kT}{hc}$ against A for relativistic case,

curve (*c*) shows $x = \epsilon_m / kT$ against A for non-relativistic case,

curve (*d*) shows $x = \epsilon_m / kT$ against A for relativistic case

In figure (2) are shown curves for the degenerate case ($1 < A < \infty$).

curve (*a*) shows $y = \frac{\lambda_m \sqrt{2mkT}}{h}$ against A for the non-relativistic case,

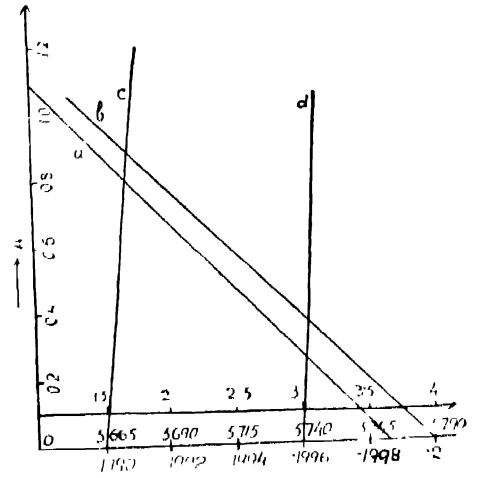

FIGURE 1

The uppermost scale x refers to curves c and d.

The middle scale y refers to curve a.

The lowermost scale z refers to curve b.

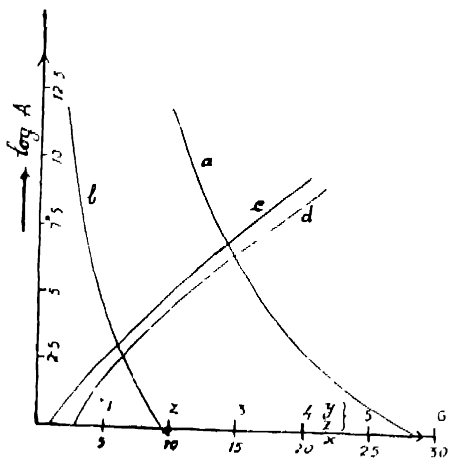

FIGURE 2

The upper scale for y, z refers to curves a and b.

The lower scale for x refers to curves c and d.

curve (b) shows $z=\dfrac{\lambda_m kT}{hc}$ against A for the relativistic case,

curve (c) shows $x=\epsilon_m/kT$ against A for the non-relativistic case,
curve (d) shows $x=\epsilon_m/kT$ against A for the relativistic case.

The ordinate in figure 1 represents A and that in figure (2) log A. The abscissa represents x, y or z as the case may be.

Table 1 shows the limits of the values of x, y and z as A varies from 0 to 1 in the non-degenerate case and from 1 to ∞ in the degenerate case.

TABLE I

Non-relativistic case $\left(\dfrac{mc^2}{kT} >> 1\right)$

	Range of ϵ_m/kT	Range of $\lambda_m \dfrac{\sqrt{2kT}}{h}$
Degenerate case, A varies from 1 to ∞	1·756 to ∞	0·5652 to 0
Non-degenerate case, A varies from 0 to 1	1·5 to 1·756	0·5774 to 0·5652

Relativistic case $\left(\frac{mc^2}{kT} << 1\right)$

	Range of ϵ_m/kT	Range of $\lambda_m kT/hc$
Degenerate case, A varies from 1 to ∞	3·13 to ∞	0·1994 to 0
Non-degenerate case, A varies from 0 to 1	3·0 to 3·13	0·2 to 0·1994

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